Promoting finite to infinite symmetries: the $3+1$-dimensional analogue of the Virasoro algebra and higher-spin fields

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Abstract

Infinite enlargements of finite pseudo-unitary symmetries are explicitly provided in this letter. The particular case of $u(2,2) \simeq so(4,2) \oplus u(1)$ constitutes a (Virasoro-like) infinite-dimensional generalization of the $3+1$-dimensional conformal symmetry, in addition to matter fields with all conformal spins. These algebras provide a new arena for integrable field models in higher dimensions; for example, Anti-de Sitter and conformal gauge theories of higher-$so(4,2)$-spin fields. A proposal for a non-commutative geometrical interpretation of space is also outlined.

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Conformal symmetry $SO(D,2)$ in D space-time dimensions is said to be finite-dimensional except for $D=2$, in which case it can be enlarged to two copies (the right- and left- moving modes) of the infinite-dimensional Witt algebra (center-less Virasoro algebra):

$$[L_m, L_n] = (m-n)L_{m+n}\,, \quad m,n \in Z. \tag{1}$$

The infinite and non-Abelian character of this algebra makes possible the exact solvability of conformally-invariant (quantum and statistical) non-linear field theories in one and two dimensions, and helps with systems in higher dimensions which, in some essential respects, are one- or two-dimensional (e.g. String Theory). There can be no doubt that a higher-dimensional counterpart of this infinite symmetry would provide novel insights into the analysis and formulation of new non-linear integrable conformal field theories and, more ambitiously, a potential starting point to formulate quantum gravity models in realistic dimensions. We shall show how such an analogue can be found inside a more general family of infinite-dimensional algebras based on pseudo-unitary symmetries.

The Witt algebra (1) can be also seen as a sort of "analytic continuation" (see e.g. [1, 2] for similar concepts) of $su(1,1) = \{L_n, |n| \leq 1\}$, that is, an extension beyond the "wedge" $|n| \leq 1$ by revoking this restriction to $n \in Z$. Analytic continuations of pseudo-unitary algebras $u(N_+, N_-)$ can be given as well and they provide, from this point of view, infinite-dimensional generalizations of the conformal symmetry in realistic dimensions for the particular case of $u(2,2) \simeq so(4,2) \oplus u(1)$.

Indeed, let us denote by $X_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, N \equiv N_+ + N_-$, the $U(N_+, N_-)$ Lie-algebra of (step) generators with commutation relations:

$$[X_{\alpha_1\beta_1}, X_{\alpha_2\beta_2}] = (\eta_{\alpha_1\beta_2}X_{\alpha_2\beta_1} - \eta_{\alpha_2\beta_1}X_{\alpha_1\beta_2})\,, \tag{2}$$

where $\eta = \mathrm{diag}(1, \overset{N_+}{\ldots}, 1, -1, \overset{N_-}{\ldots}, -1)$ is used to raise and lower indices (see a clarifying Note at the end of the paper). For example, for $u(1,1)$ we have: $L_1 = X_{12}, L_{-1} = X_{21}, L_0 = \frac{1}{2}(X_{22} + X_{11})$. It is of importance that the commutation relations (2) can also be written as

$$\left[L^I_m, L^J_n\right] = \eta^{\alpha\beta}(J_\alpha m_\beta - I_\alpha n_\beta)L^{I+J-\delta_\alpha}_{m+n}\,, \tag{3}$$

where the the lower index m of L now symbolizes a integral upper-triangular $N \times N$ matrix and the upper (generalized spin) index $I \equiv (I_1, \ldots, I_N)$ represents a N-dimensional vector which, in

general, can be taken to lie on a half-integral lattice; we denote by $m_\alpha \equiv \sum_{\beta>\alpha} m_{\alpha\beta} - \sum_{\beta<\alpha} m_{\beta\alpha}$ and $\delta_\alpha \equiv (\delta_{\alpha,1}, \ldots, \delta_{\alpha,N})$. Note that, the generators L^I_m are labeled by $N + N(N-1)/2 = N(N+1)/2$ indices, in the same way as wave functions ψ^I_m in the carrier space of irreps of $U(N)$. There are many possible ways of embedding the $u(N_+, N_-)$ generators (2) inside (3), as there are also many possible choices of $su(1,1)$ inside (1). However, a "canonical" choice is

$$X_{\alpha\beta} \equiv L^{\delta_\alpha}_{x_{\alpha\beta}}\,, \quad x_{\alpha\beta} \equiv \mathrm{sign}(\beta-\alpha) \sum_{\sigma=\mathrm{Min}(\alpha,\beta)}^{\mathrm{Max}(\alpha,\beta)-1} x_{\sigma,\sigma+1}\,, \tag{4}$$

where $x_{\sigma,\sigma+1}$ denotes an upper-triangular matrix with zero entries, except for one at the $(\sigma,\sigma+1)$-position, that is $(x_{\sigma,\sigma+1})_{\mu\nu} = \delta_{\sigma,\mu}\delta_{\sigma+1,\nu}$ (we set $x_{\alpha\alpha} \equiv 0$). For example, the $u(1,1)$ Lie-algebra generators correspond to:

$$X_{12} = L^{(1,0)}_{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \quad X_{21} = L^{(0,1)}_{\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}}, \quad X_{11} = L^{(1,0)}_{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}, \quad X_{22} = L^{(0,1)}_{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}. \tag{5}$$

Relaxing the restrictions $m = x_{\alpha\beta}$ in (4) to arbitrary integral upper-triangular matrices m leads to the following infinite-dimensional algebra (as can be seen from (3))

$$\left[L^{\delta_\alpha}_m, L^{\delta_\beta}_n\right] = m^\beta L^{\delta_\alpha}_{m+n} - n^\alpha L^{\delta_\beta}_{m+n}\,, \tag{6}$$

which we shall denote by $\mathcal{L}^{(1)}_\infty(u(N_+,N_-))$. It is easy to see that, for $u(1,1)$, this "analytic continuation" leads to two Virasoro sectors: $L_{m_{12}} \equiv L^{(1,0)}_m$, $\bar{L}_{m_{12}} \equiv L^{(0,1)}_m$. Its $3+1$ dimensional counterpart $\mathcal{L}^{(1)}_\infty(u(2,2))$ contains four non-commuting Virasoro-like sectors $\mathcal{L}^{(1_\alpha)}_\infty(u(2,2)) = \{L^{\delta_\alpha}_m\}$, $\alpha = 1, \ldots, 4$ which, in their turn, hold three genuine Virasoro sectors for $m = ku_{\alpha\beta}$, $k \in Z$, $\alpha < \beta = 2, \ldots, 4$, where $u_{\alpha\beta}$ denotes an upper-triangular matrix with components $(u_{\alpha\beta})_{\mu\nu} = \delta_{\alpha,\mu}\delta_{\beta,\nu}$. In general, $\mathcal{L}^{(1)}_\infty(u(N_+,N_-))$ contains $N(N-1)$ distinct and non-commuting Virasoro sectors, and holds $u(N_+,N_-)$ as the *maximal finite-dimensional subalgebra*.

The algebra $\mathcal{L}^{(1)}_\infty(u(N_+,N_-))$ can be seen as the *minimal* infinite continuation of $u(N_+,N_-)$ representing the diffeomorphism algebra diff(N) of the corresponding N-dimensional manifold (locally the Minkowski space-time for $u(2,2)$). Indeed, the algebra (6) formally coincides with the algebra of vector fields $L^\mu_{f(y)} = f(y)\frac{\partial}{\partial y_\mu}$, where $y = (y_1, \ldots, y_N)$ denotes a local system of coordinates and $f(y)$ can be expanded in a plane wave basis, such that $L^\mu_{\vec{m}} = e^{im^\alpha y_\alpha}\frac{\partial}{\partial y_\mu}$ constitutes a basis of vector fields for the so called generalized Witt algebra [3], of which there

are studies about its representations (see e.g. [4, 5, 6]). Note that, for us, the N-dimensional lattice vector $\vec{m} = (m_1, \ldots, m_N)$ is constrained by $\sum_{\alpha=1}^{N} m_\alpha = 0$ (see the definition of m_α in paragraph after Eq. 3), which introduces some novelties as regards the Witt algebra. In fact, the algebra (6) can be split into one "temporal" piece, constituted by an Abelian ideal generated by $\check{L}^N_m \equiv \eta_{\alpha\alpha} L^{\delta_\alpha}_m$, and a "residual" symmetry generated by the spatial diffeomorphisms

$$\check{L}^j_m \equiv \eta_{jj} L^{\delta_j}_m - \eta_{j+1,j+1} L^{\delta_{j+1}}_m,\ j = 1, \ldots, N-1 \text{ (no sum on } j), \tag{7}$$

which act semi-directly on the temporal part. More precisely, the commutation relations (6) in this new basis adopt the following form:

$$\begin{aligned} \left[\check{L}^j_m, \check{L}^k_n\right] &= \check{m}^k \check{L}^j_{m+n} - \check{n}^j \check{L}^k_{m+n}\,, \\ \left[\check{L}^j_m, \check{L}^N_n\right] &= -\check{n}^j \check{L}^N_{m+n}\,, \\ \left[\check{L}^N_m, \check{L}^N_n\right] &= 0\,, \end{aligned} \tag{8}$$

where $\check{m}_k \equiv m_k - m_{k+1}$. Only for $N = 2$, the last commutator admits a central extension of the form $\sim n_{12}\delta_{m+n,0}$ compatible with the rest of commutation relations (8). This result amounts to the fact that the (unconstrained) diffeomorphism algebra diff(N) does not admit any non-trivial central extension except when $N = 1$ [7].

Additionally, after the restriction $I = \delta_\alpha$ in (4) is also relaxed to arbitrary half-integral lattice vectors I, the commutation relations (3) define a *higher-$u(N_+, N_-)$-spin algebra* $\mathcal{L}_\infty(u(N_+, N_-))$ (in a sense similar to that of Ref. [8]), which contains $\mathcal{L}^{(1)}_\infty(u(N_+, N_-))$ as a subalgebra as well as all *matter fields* L^I_m with all $u(N_+, N_-)$-spins I.

The quantization procedure for the algebra (3) entails unavoidable renormalizations (mainly due to ordering problems) and central extensions like:

$$\left[\hat{L}^I_m, \hat{L}^J_n\right] = \hbar\eta^{\alpha\beta}(J_\alpha m_\beta - I_\alpha n_\beta)\hat{L}^{I+J-\delta_\alpha}_{m+n} + O(\hbar^3) + \hbar^{(\sum_{\alpha=1}^{N} I_\alpha + J_\alpha)} c^{(I,J)}(m)\delta_{m+n,0}\hat{1}\,, \tag{9}$$

where $\hat{1} \sim \hat{L}^0_0$ denotes a central generator and $c^{(I,J)}(m)$ are central charges. The higher order terms $O(\hbar^3)$ can be captured in a classical construction by extending the classical (Poisson-Lie) bracket (3) to the Moyal bracket (see [9] for more information on Moyal deformation).

Central extensions provide the essential ingredient required to construct invariant geometric action functionals on coadjoint orbits of the corresponding groups. When applied to the infinite continuation (9) of $u(2, 2)$, this would lead to Wess-Zumino-Witten-like models for *induced*

gravities in $3+1$ *dimensions*, as happens for the Virasoro and $\mathcal{W}$ algebras in $1+1D$ (see e.g. [10]). The minimal coupling to matter could be done just by adding the *full* set of conformal fields $\hat{L}^I_m$ with all $u(2,2)$-spins $I=(I_1,I_2,I_3,I_4)$.

The higher-$u(N_+,N_-)$-spin algebras (9) provide the arena for new non-linear integrable quantum field models in higher dimensions, the particular cases $u(2,2)\simeq so(4,2)\oplus u(1)$ and $u(4)\simeq so(6)\oplus u(1)$ being also a potential guiding principle towards the still unknown "M-theory".

An additional point which is also worth-mentioning, is that the algebra (9) is actually a member of a N-parameter family $\tilde{\mathcal{L}}_{\vec\rho}(u(N_+,N_-))$, $\vec\rho\equiv(\rho_1,\dots,\rho_N)$ of non-isomorphic algebras of $U(N_+,N_-)$ tensor operators (see [9]), the classical limit $\hbar\to 0,\rho_\alpha\to\infty$ corresponding to the classical (Poisson-Lie) algebra $\mathcal{L}_\infty(u(N_+,N_-))$ with commutation relations (3). A very interesting feature of $\tilde{\mathcal{L}}_{\vec\rho}(u(N_+,N_-))$ is that it *collapses* to $\mathrm{Mat}_d(C)$ (the full matrix algebra of $d\times d$ complex matrices) whenever the (complex) parameters ρ_α coincide with the eigenvalues q_α of the Casimir operators C_α of $u(N_+,N_-)$ in a d-dimensional irrep $D_{\vec q}$ of $u(N_+,N_-)$. This fact can provide finite (d-points) 'fuzzy' or 'cellular' descriptions of the non-commutative counterpart of AdS_5 (a desirable property as regards finite models of quantum gravity) when applying the ideas of *non-commutative geometry* (see e.g. [11]) to $\tilde{\mathcal{L}}_{\vec\rho}(u(2,2))$.

Let us illustrate briefly this phenomenon with the simple example of the algebra $C^\infty(T^2)$ of smooth functions $L_{\vec m}=e^{\frac{2\pi i}{\ell}\vec m\cdot\vec y}$ on a torus T^2, where $\vec y=(y_1,y_2)$ denote the coordinates (modulo ℓ) and $\vec m=(m_1,m_2)$ are a pair of integers. The ordinary product of functions $L_{\vec m}\cdot L_{\vec n}=L_{\vec m+\vec n}$ defines $C^\infty(T^2)$ as an associative and commutative algebra. Yet, it must be emphasized that this product is actually a limiting case of a more fundamental (quantum) associative and *non-commutative* $\star$-product $\hat L_{\vec m}\star\hat L_{\vec n}=e^{2\pi i\frac{\lambda^2}{\ell^2}\vec m\times\vec n}\hat L_{\vec m+\vec n}$, where $\vec m\times\vec n\equiv m_1n_2-m_2n_1$, λ is a parameter with dimensions of length (e.g. the Planck length $\sqrt{G\hbar}$) and $\hat L_{\vec m}$ denotes a *symbol* (the non-commutative counterpart of the function $L_{\vec m}$). The commutator of two symbols is defined as:

$$\left[\hat L_{\vec m},\hat L_{\vec n}\right]\equiv\hat L_{\vec m}\star\hat L_{\vec n}-\hat L_{\vec n}\star\hat L_{\vec m}=2i\sin\left(2\pi\frac{\lambda^2}{\ell^2}\vec m\times\vec n\right)\hat L_{\vec m+\vec n}\,. \tag{10}$$

Note that, when the surface of the torus ℓ^2 contains an integer number q of times the *minimal cell* λ^2 (that is, $\ell^2=q\lambda^2$), the infinite-dimensional algebra (10) collapses to a finite-dimensional matrix algebra: the Lie algebra of the unitary group $U(q/2)$ for q even or $SU(q)\times U(1)$ for q odd

(see [12]). In fact, taking the quotient in (10) by the equivalence relation $\hat{L}_{\vec{m}+q\vec{a}} \sim \hat{L}_{\vec{m}}$, $\forall \vec{a} \in Z \times Z$, it can be seen that the following identification $\hat{L}_{\vec{m}} = \sum_k e^{\frac{2\pi i}{q} m_1 k} X_{k,k+m_2}$ implies a change of basis in the step-operator algebra (2) of $U(q)$.

Thinking of $\rho = \frac{\ell^2}{\lambda\!\!\!^{-}{}^2}$ as a 'density of points', we can conclude that: for the critical values $\rho_c = q \in Z$, the Lie algebra (10) —which we denote by $C^\star_\rho(T^2)$— is *finite*; that is, the quantum analogue of the torus has a 'finite number q of points'. It is in this sense that we talk about a 'cellular structure of the space'. Moreover, given the basic commutator $[y_1, y_2] = -i\lambda\!\!\!^{-}{}^2/\pi$, this cellular structure is a consequence of the absence of localization expressed by the Heisenberg uncertainty relation $\Delta y_1 \Delta y_2 \geq \lambda\!\!\!^{-}{}^2/(2\pi)$.

In the (classical) limit of large number of points $\rho \to \infty$ and $\lambda\!\!\!^{-} \to 0$ (such that $\ell^2 = \rho\lambda\!\!\!^{-}{}^2$ remains finite) we recover the original (commutative) geometry on the torus. For example, it is easy to see that

$$\lim_{\substack{\rho \to \infty \\ \lambda\!\!\!^{-} \to 0}} \frac{i\pi}{\lambda\!\!\!^{-}{}^2} \left[\hat{L}_{\vec{m}}, \hat{L}_{\vec{n}}\right] = \frac{4\pi^2}{\ell^2} \vec{n} \times \vec{m}\, \hat{L}_{\vec{m}+\vec{n}} \tag{11}$$

coincides with the (classical) Poisson bracket $\{L_{\vec{m}}, L_{\vec{n}}\} = \Upsilon_{jk} \frac{\partial L_{\vec{m}}}{\partial y_j} \frac{\partial L_{\vec{n}}}{\partial y_k}$ of functions in $C^\infty(T^2)$, where $\Upsilon_{2\times 2} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ denotes the symplectic form on the torus.

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Note added: when working in the complexification of $u(N_+, N_-)$, the Lie algebras $u(N_+, N_-) = \{X_{\alpha\beta}\}$ and $u(N) = \{X^\gamma_\alpha\}$ are related by $X_{\alpha\beta} = \eta_{\beta\gamma} X^\gamma_\alpha$; however, this relation does not mean that $u(N_+, N_-)$ and $u(N)$ are isomorphic as such. Indeed, let us take the simple case of

$$G(\kappa) = \left\{ g = \begin{pmatrix} z_1 & \kappa \bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, z_i, \bar{z}_i \in C / \det(g) = |z_1|^2 - \kappa |z_2|^2 = 1 \right\}, \tag{12}$$

which reduces to $G(1) = SU(1,1)$ and $G(-1) = SU(2)$ for $\kappa = \pm 1$. Let us choose the following system of complex coordinates: $z \equiv \frac{z_2}{z_1}$, $\bar{z} \equiv \frac{\bar{z}_2}{\bar{z}_1}$, $\xi \equiv \frac{z_1}{|z_1|}$; they correspond to the standard

stereographic projection coordinates $z, \bar{z}$ of the sphere (resp. hyperboloid) on the complex plane C (resp. unit disk) for $SU(2)$ (resp. $SU(1,1)$), and ξ is the Cartan phase. After a little bit of algebra, one can see that there is a change of basis between both ($su(2)$ and $su(1,1)$) *abstract* Lie algebras of left-invariant vector fields X of the form: $X_z^{SU(2)} = X_z^{SU(1,1)}$, $X_{\bar{z}}^{SU(2)} = -X_{\bar{z}}^{SU(1,1)}$, $X_\xi^{SU(2)} = X_\xi^{SU(1,1)}$ (see [13] for more details); however, as stated above, it does not mean that $su(2)$ and $su(1,1)$ are isomorphic!. As abstract algebras, the main difference arises from the point of view of representations, when $(X_z^{SU(1,1)})^\dagger = X_{\bar{z}}^{SU(1,1)}$ whereas $(X_z^{SU(2)})^\dagger = -X_{\bar{z}}^{SU(2)}$.

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